On certain Groups of Relations satisfied by the Quadruple Theta-Functions.

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If m_1, m_2, m_3, m_4 denote even integers, positive or negative, and if we write

$$\begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}$$

$$= \tfrac{1}{4}(a_{11}, a_{12}, \ldots a_{44})(m_1+\alpha_1, \ldots m_4+\alpha_4)^2$$

$$+ \tfrac{1}{2}\pi i\,[(m_1+\alpha_1)(u_1+\beta_1) + \ldots + (m_4+\alpha_4)(u_4+\beta_4)]$$

we have for the definition of the quadruple theta-functions the equation

$$\vartheta\begin{pmatrix} \alpha_1\,\alpha_2\,\alpha_3\,\alpha_4 \\ \beta_1\,\beta_2\,\beta_3\,\beta_4 \end{pmatrix}(u_1 u_2 u_3 u_4) =$$

$$\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4} \exp. \begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}$$

or briefly $$\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(u) = \Sigma \exp. \begin{pmatrix} m+\alpha \\ u+\beta\end{pmatrix},$$

where the summations extend over all positive and negative even integer values of m_1, m_2, m_3 and m_4. Suppose now we write

$$a_{11} = \log q_1,\quad a_{22} = \log q_2,\quad a_{33} = \log q_3,\quad a_{44} = \log q_4,$$

$$a_{12} = \log q_{12},\quad a_{13} = \log q_{13},\quad a_{14} = \log q_{14},\quad a_{23} = \log q_{23}, \text{ etc.}$$

and also $$u_1 = \frac{v_1}{K_1},\quad u_2 = \frac{v_2}{K_2},\quad u_3 = \frac{v_3}{K_3},\quad u_4 = \frac{v_4}{K_4},$$

and replace the m_1, m_2, m_3, m_4 by $2m_1$, etc.; then the summations will extend over all positive and negative values of the m's, and it is easy to see that we have

$$\vartheta\begin{pmatrix} \alpha_1\,\alpha_2\,\alpha_3\,\alpha_4 \\ \beta_1\,\beta_2\,\beta_3\,\beta_4 \end{pmatrix}(u_1 u_2 u_3 u_4) =$$

$$\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4} (-)^{\Sigma m\beta} \exp. \Big\{ \tfrac{1}{4}\,[(2m_1+\alpha_1)^2 \log q_1 + \ldots + (2m_4+\alpha_4)^2 \log q_4$$

$$+ 2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}$$

$$+ \ldots + 2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}]$$

$$+ \frac{i\pi}{2}\left((2m_1+\alpha_1)\frac{v_1}{K_1} + \ldots + (2m_4+\alpha_4)\frac{v_4}{K_4}\right)\Big\}.$$

The true periods are obviously

v_1	$4K_1$	0	0	0
v_2	0	$4K_2$	0	0
v_3	0	0	$4K_3$	0
v_4	0	0	0	$4K_4$

The quasi-periods are easily written down, but it is not worth while to give them here.

Write the exponent in the following manner:

$$\tfrac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K}(2m_1+\alpha_1)$$
$$+\left\{\tfrac{1}{4}(2m_2+\alpha_2)^2\log q_2+\tfrac{1}{4}(2m_3+\alpha_3)^2\log q_3+\tfrac{1}{4}(2m_4+\alpha_4)^2\log q_4\right.$$
$$+2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}+2(2m_1+\alpha_1)(2m_3+\alpha_3)\log q_{13}$$
$$+2(2m_1+\alpha_1)(2m_4+\alpha_4)\log q_{14}$$
$$+2(2m_2+\alpha_2)(2m_3+\alpha_3)\log q_{23}+2(2m_2+\alpha_2)(2m_4+\alpha_4)\log q_{24}$$
$$+2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}$$
$$\left.+\frac{i\pi v_2}{2K_2}(2m_2+\alpha_2)+\frac{i\pi v_3}{2K_3}(2m_3+\alpha_3)+\frac{i\pi v_4}{2K_4}(2m_4+\alpha_4)\right\}.$$

Write now

$$\log q_{12}=\frac{i\pi}{2K_2}\log p_{12},\quad \log q_{13}=\frac{i\pi}{2K_3}\log p_{13},\quad \log q_{14}=\frac{i\pi}{2K_4}\log p_{14}$$

and

$$v_2=w_2-\frac{2m_1+\alpha_1}{2}\log p_{12},\quad v_3=w_3-\frac{2m_1+\alpha_1}{2}\log p_{13},\quad v_4=w_4-\frac{2m_1+\alpha_1}{2}\log p_{14}.$$

Taking the terms which contain $\log q_{12}$ and v_2 and combine them after making these substitutions and we have

$$\tfrac{1}{4}\cdot 2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}+\frac{i\pi v_2}{2K_2}(2m_2+\alpha_2)$$
$$=\tfrac{1}{4}\cdot 2(2m_1+\alpha_1)(2m_2+\alpha_2)\frac{i\pi}{2K_2}\log p_{12}+(2m_2+\alpha_2)\left(\frac{i\pi w_2}{2K_2}-\frac{i\pi}{2K_2}\cdot\frac{2m_1+\alpha_1}{2}\log p_{12}\right)$$
$$=\frac{i\pi w_2}{2K_2}(2m_2+\alpha_2).$$

Similar results are obtained for the terms containing $\log q_{13}$ and v_3, $\log q_{14}$ and v_4. Combining all these and the new exponent is:

$$\tfrac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K_1}(2m_1+\alpha_1)$$

$$+\Big\{\tfrac{1}{4}(2m_2+\alpha_2)^2\log q_2+\tfrac{1}{4}(2m_3+\alpha_3)^2\log q_3+\tfrac{1}{4}(2m_4+\alpha_4)^2\log q_4$$
$$+2(2m_2+\alpha_2)(2m_3+\alpha_3)\log q_{23}+2(2m_2+\alpha_2)(2m_4+\alpha_4)\log q_{24}$$
$$+2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}$$
$$+\frac{i\pi w_2}{2K_2}(2m_2+\alpha_2)+\frac{i\pi w_3}{2K_3}(2m_3+\alpha_3)+\frac{i\pi w_4}{2K_4}(2m_4+\alpha_4)\Big\}.$$

The terms in $\{\ \}$ obviously form the exponent of a triple theta-function, viz.

$$\vartheta\begin{pmatrix}\alpha_2\,\alpha_3\,\alpha_4\\ \beta_2\,\beta_3\,\beta_4\end{pmatrix}(w_2w_3w_4),\ \text{or simply}\ \vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(w)$$

understanding that here the suffixes are 2, 3, 4. We have then finally

$$1.\ \vartheta\begin{pmatrix}\alpha_1\alpha_2\alpha_3\alpha_4\\ \beta_1\beta_2\beta_3\beta_4\end{pmatrix}(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1}e^{\frac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K_1}(2m_1+\alpha_1)}.\,\vartheta\begin{pmatrix}\alpha_2\alpha_3\alpha_4\\ \beta_2\beta_3\beta_4\end{pmatrix}(w_2w_3w_4)$$

For brevity write this in the form

$$2.\qquad \Theta_4(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1},\ q_1^{\left(m_1+\frac{\alpha_1}{2}\right)^2}e^{\frac{i\pi v_1}{K_1}\left(m_1+\frac{\alpha_1}{2}\right)}.\,\Theta_3(w_2w_3w_4),$$

or substituting for w_2, w_3, w_4 their values

$$3.\quad \Theta_4(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1}q_1^{\left(m_1+\frac{\alpha_1}{2}\right)^2}e^{\frac{i\pi v_1}{K_1}\left(m_1+\frac{\alpha_1}{2}\right)}$$
$$\Theta_3\left(v_2+\frac{2m_1+\alpha_1}{2}\log p_{12},\ v_3+\frac{2m_1+\alpha_1}{2}\log p_{13},\ v_4+\frac{2m_1+\alpha_1}{2}\log p_{14}\right),$$

the summation extending from $m_1=-\infty$ to $m_1=+\infty$. Expand this by giving to m_1 all of its values and grouping together the terms corresponding to equal positive and negative values of m_1,

$$4.\qquad \Theta_4(v_1v_2v_3v_4)=q_1^{\frac{\alpha_1^2}{4}}e^{\frac{i\pi v_1}{K_1}\cdot\frac{\alpha_1}{2}}\cdot\Theta_3\left(v_2+\frac{\alpha_1}{2}\log p_{12},\ v_3+\frac{\alpha_1}{2}\log p_{13},\ v_4+\frac{\alpha_1}{2}\log p_{14}\right)$$
$$+(-)^{\beta_1}q_1^{\frac{(2+\alpha_1)^2}{4}}\Big\{\cos(2+\alpha_1)\tau\Big[\Theta_3\left(v_2+\frac{2+\alpha_1}{2}\log p_{12}\ldots\right)$$
$$+q^{-4\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\left(v_2-\frac{2-\alpha_1}{2}\log p_{12}\ldots\right)\Big]$$
$$+i\sin(2+\alpha_1)\tau\Big[\Theta_3\left(v_2+\frac{2+\alpha_1}{2}\log p_{12}\ldots\right)$$
$$-q^{-4\alpha_1}\cos(2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\left(v_2-\frac{2-\alpha_1}{2}\log p_{12}\ldots\right)\Big]\Big\}$$
$$+(-)^{2\beta_1}q^{\frac{(4+\alpha_1)^2}{4}}\Big\{\cos(4+\alpha_1)\tau\Big[\Theta_3\left(v_2+\frac{4+\alpha_1}{2}\log p_{12}\ldots\right)$$
$$+q^{-16\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\left(v_2-\frac{4-\alpha_1}{2}\log p_{12}\ldots\right)\Big]$$
$$+i\sin(4+\alpha_1)\tau\Big[\Theta_3\left(v_2+\frac{4+\alpha_1}{2}\log p_{12}\ldots\right)$$
$$-q^{-16\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\left(v_3-\frac{4-\alpha_1}{2}\log p_{12}\ldots\right)\Big]\Big\}$$
$$+\text{etc.}$$

where for brevity I have written $\tau_1 = \frac{\pi v_1}{2K_1}$.

The single theta-function $\vartheta\binom{\alpha_1}{\beta_1}(v_1)$ on being expanded takes the form

$$5.\ \vartheta\binom{\alpha_1}{\beta_1}(v_1) = q_1^{\frac{\alpha_1^2}{4}} e^{\alpha_1\tau_1} + (-)^{\beta_1} q_1^{\frac{(2+\alpha_1)^2}{4}} \{\cos(2+\alpha_1)\tau_1[1+q_1^{-4\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]$$
$$+ i\sin(2+\alpha_1)\tau_1[1-q_1^{-4\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]\}$$
$$+ (-)^{2\beta_1} q_1^{\frac{(4+\alpha_1)^2}{4}} \{\cos(4+\alpha_1)\tau_1[1+q^{-16\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]$$
$$+ i\sin(4+\alpha_1)\tau_1[1-q_1^{-16\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]\}$$
$$+ \text{etc.}$$

Giving $\binom{\alpha_1}{\beta_1}$ the values $\binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$, $\binom{1}{1}$ this general form gives the four known equations

$$\vartheta\binom{0}{0}(v_1) = 1 + 2q_1 \cos 2\frac{\pi v_1}{2K_1} + 2q_1^4 \cos 4\frac{\pi v_1}{2K_1} + 2q_1^9 \cos 6\frac{\pi v_1}{2K_1} + \dots$$
$$\vartheta\binom{1}{0}(v_1) = 1 - 2q_1 \cos 2\frac{\pi v_1}{2K_1} + 2q_1^4 \cos 4\frac{\pi v_1}{2K_1} - 2q_1^9 \cos 6\frac{\pi v_1}{2K_1} + \dots$$
$$\vartheta\binom{0}{1}(v_1) = 2q_1^{\frac{1}{4}} \cos \frac{\pi v_1}{2K_1} + 2q_1^{\frac{9}{4}} \cos 3\frac{\pi v_1}{2K_1} + 2q_1^{\frac{25}{4}} \cos 5\frac{\pi v_1}{2K_1} + \dots$$
$$-i\vartheta\binom{1}{1}(v_1) = 2q_1^{\frac{1}{4}} \sin \frac{\pi v_1}{2K_1} - 2q_1^{\frac{9}{4}} \cos 3\frac{\pi v_1}{2K_1} + 2q_1^{\frac{25}{4}} \sin 5\frac{\pi v_1}{2K_1} + \dots$$

For convenience write $\vartheta\binom{\alpha_1}{\beta_1}(v_1)$ as simply $\Theta_1(v_1)$ and bear in mind that

$$\Theta_4(v_1 v_2 v_3 v_4) \equiv \vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1 v_2 v_3 v_4)$$

and
$$\Theta_3(v_2 v_3 v_4) \equiv \vartheta\begin{pmatrix}\alpha_2\,\alpha_3\,\alpha_4\\ \beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_2 v_3 v_4).$$

Now by aid of 5, equation 4 may be thrown into the form

$$6.\qquad \Theta_4(v_1, v_2, v_3, v_4) = \Theta_3(v_2, v_3, v_4).\Theta_1(v_1)$$
$$-\frac{K_1}{\pi}\frac{d\Theta_1(v_1)}{dv_1}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$+\frac{1}{2}\left(\frac{K_1}{\pi}\right)^2\frac{d^2\Theta_1(v_1)}{dv_1^2}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$-\frac{1}{3}\left(\frac{K_1}{\pi}\right)^3\frac{d^3\Theta(v_1)}{dv_1^3}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$+ \text{etc.},$$

or symbolically this may be written,

$$7.\qquad \Theta_4 = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\log q_{1n}.K_1K_n\frac{d^2}{dv_1 dv_n}}.\Theta_1.\Theta_3,$$

and obviously by transforming Θ_3 in the same manner we have

8. $$\Theta_4 = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\sum_{l=1}^{l=4}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \prod_{l=1}^{l=4} \Theta_{1_l}$$

where $$\prod_{l=1}^{l=4} \Theta_{1_l} = \Theta_1(v_1)\Theta_2(v_2)\Theta_3(v_3)\Theta_4(v_4),$$

and in the double summation n and l are always to have different values. The generalization of 8 for the p-tuple theta-functions is

9. $$\Theta_p = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=p}\sum_{l=1}^{l=p}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \prod_{l=1}^{l=p} \Theta_{1_l}.$$

Equation 8 written out in full is

10. $$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1 v_2 v_3 v_4)$$
$$= e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\sum_{l=1}^{l=4}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \vartheta\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}(v_1)\,\vartheta\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}(v_2)\,\vartheta\begin{pmatrix}\alpha_3\\ \beta_3\end{pmatrix}(v_3)\,\vartheta\begin{pmatrix}\alpha_4\\ \beta_4\end{pmatrix}(v_4).$$

The quantities K_1, K_2, K_3, K_4 may be taken as the complete elliptic integrals of the first kind corresponding to moduli k_1, k_2, k_3, k_4; similarly E_1, E_2, E_3, E_4 may be taken as the complete elliptic integrals of the second kind corresponding to the same moduli. Now writing $\vartheta = \vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(v)$ we have (Cayley's Elliptic Functions, page 227),

$$\frac{d^2\vartheta}{dv^2} - 2v\left(k'^2 - \frac{E}{K}\right)\frac{d\vartheta}{dv} + 2kk'^2\frac{d\vartheta}{dk} = 0,$$

or letting i denote either 1, 2, 3, or 4,

11. $$\frac{d^2\vartheta_i}{dv_i^2} - 2v_i\left(k_i'^2 - \frac{E_i}{K_i}\right)\frac{d\vartheta_i}{dv_i} + 2k_i k_i'^2\frac{d\vartheta_i}{dk_i} = 0.*$$

Now (Cayley, page 102) we have

12. $$\frac{dK_i}{dk_i} = \frac{1}{k_i k_i'^2}(E_i - k_i'^2 K_i),$$

so that 11 may be written in the form

13. $$\frac{d^2\vartheta_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\vartheta_i}{dv_i} + 2k_i k_i'^2\frac{d\vartheta_i}{dk_i} = 0.$$

Differentiating this s times with respect to v_i and we have

14. $$\frac{d^2}{dv_i^2}\frac{d^s\vartheta_i}{dv_i^s} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d}{dv_i}\frac{d^s\vartheta_i}{dv_i^s} + 2k_i k_i'^2\frac{d}{dk_i}\frac{d^s\vartheta_i}{dv_i^s} = 0.$$

Now so far as x_i and k_i are concerned the general term in 8 is a numerical multiple of

$$K_i^s\frac{d^s\vartheta_i}{dv_i^s},\ \text{say } \phi_i,$$

then

15. $$\frac{d^2\varphi_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\varphi_i}{dv_i} = K_i^s\left\{\frac{d^2}{dv_i^2}\frac{d^s\vartheta_i}{dv_i^s} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\vartheta_i}{dv_i}\right\},$$

*For what immediately follows I am indebted to Mr. Forsyth's paper on Theta Functions: *Phil. Trans.*, 1882.

and

16. $$\frac{d\varphi_i}{dk_i} = K_i^s \left\{ \frac{d}{dk_i} \frac{d^s \vartheta_i}{dv_i^s} + \frac{s}{K_i} \frac{dK_i}{dk_i} \frac{d^s \vartheta_i}{dv_i^s} \right\}.$$

Hence

17. $$\frac{d^2 \varphi_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i} \frac{dK_i}{dk_i} v_i \frac{d\varphi_i}{dv_i} + 2k_i k_i'^2 \frac{d\varphi_i}{dk_i} = 0,$$

and now since Θ_4 is the sum of the quantities ϕ_i we have

18. $$\frac{d^2 \theta_4}{dv_i^2} + \frac{2k_i k_i'^2}{K_i} \frac{dK_i}{dk_i} v_i \frac{d\theta_4}{dv_i} + 2k_i k_i'^2 \frac{d\theta_4}{dk_i} = 0,$$

or

19. $$\frac{d^2 \theta_4}{dv_i^2} - 2v_i \left(k_i'^2 - \frac{E_i}{K_i} \right) \frac{d\theta_4}{dv_i} + 2k_i k_i'^2 \frac{d\theta_4}{dk_i} = 0.$$

There are of course four equations of this type. Now from 8 we have

20. $$\frac{d\theta_4}{dq_{nl}} = -\frac{2}{\pi^2} K_n K_l \frac{d^2}{dv_n dv_l} \cdot \frac{1}{q_{nl}} \cdot \Theta_4,$$

and consequently

21. $$q_{nl} \frac{d\theta_4}{dq_{nl}} + \frac{2K_n K_l}{\pi^2} \frac{d^2 \theta_4}{dv_n dv_l} = 0.$$

There are obviously six equations of this type. The quadruple theta-functions therefore satisfy in all ten different equations of the second order; written out in full these are

I

$$\begin{aligned}
&\frac{d^2 \theta_4}{dv_1^2} - 2v_1 \left(k_1'^2 - \frac{E_1}{K_1} \right) \frac{d\theta_4}{dv_1} + 2k_1 k_1'^2 \frac{d\theta_4}{dk_1} = 0 \\
&\frac{d^2 \theta_4}{dv_2^2} - 2v_2 \left(k_2'^2 - \frac{E_2}{K_2} \right) \frac{d\theta_4}{dv_2} + 2k_2 k_2'^2 \frac{d\theta_4}{dk_2} = 0 \\
&\frac{d^2 \theta_4}{dv_3^2} - 2v_3 \left(k_3'^2 - \frac{E_3}{K_3} \right) \frac{d\theta_4}{dv_3} + 2k_3 k_3'^2 \frac{d\theta_4}{dk_3} = 0 \\
&\frac{d^2 \theta_4}{dv_4^2} - 2v_4 \left(k_4'^2 - \frac{E_4}{K_4} \right) \frac{d\theta_4}{dv_4} + 2k_4 k_4'^2 \frac{d\theta_4}{dk_4} = 0.
\end{aligned}$$

II

$$\begin{aligned}
&q_{12} \frac{d\theta_4}{dq_{12}} + \frac{2K_1 K_2}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_2} = 0 \\
&q_{13} \frac{d\theta_4}{dq_{13}} + \frac{2K_1 K_3}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_3} = 0 \\
&q_{14} \frac{d\theta_4}{dq_{14}} + \frac{2K_1 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_4} = 0 \\
&q_{23} \frac{d\theta_4}{dq_{23}} + \frac{2K_2 K_3}{\pi^2} \frac{d^2 \theta_4}{dv_2 dv_3} = 0 \\
&q_{24} \frac{d\theta_4}{dq_{24}} + \frac{2K_2 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_2 dv_4} = 0 \\
&q_{34} \frac{d\theta_4}{dq_{34}} + \frac{2K_3 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_3 dv_4} = 0.
\end{aligned}$$

These equations can be obtained at once from the general definition of Θ_4; this is

22. $$\Theta_4, = \vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1v_2v_3v_4), = \sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}(-)^{\Sigma m\beta}\, q_1^{\frac{(2m_1+\alpha_1)^2}{4}}\dots q_4^{\frac{(2m_4+\alpha_4)^2}{4}}$$
$$\times\, q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}}\dots q_{34}^{\frac{(2m_3+\alpha_3)(2m_4+\alpha_4)}{2}}$$
$$\times\, e^{\frac{i\pi}{2}\left\{(2m_1+\alpha_1)\frac{v_1}{K_1}+\dots+(2m_4+\alpha_4)\frac{v_4}{K_4}\right\}}.$$

The general term in Θ_4 is a multiple of

$$q_i^{\frac{(2m_i+\alpha_i)^2}{4}}\,.\,e^{\frac{i\pi v}{2K_i}(2m_i+\alpha_i)} =, \text{ say } \mathfrak{R}_i,$$

the coefficient of $\mathfrak{R}_i$ being independent of v_i and k_i. Now

23. $$\frac{d}{dq_i}\mathfrak{R} = \frac{(2m_i+\alpha_i)^2}{4}\mathfrak{R}_i - \frac{i\pi v_i(2m_i+\alpha_i)}{2K_i^2}\,\frac{dK_i}{dp_i}\mathfrak{R}_i,$$

24. $$\frac{d^2}{dv_i^2}\mathfrak{R}_i = -\frac{(2m_i+\alpha_i)^2\pi^2}{4K_i^2}\mathfrak{R}_i,$$

25. $$v_i\frac{d}{dv_i}\mathfrak{R}_i = \frac{i\pi v_i(2m_i+\alpha_i)}{2K_i}\mathfrak{R}_i;$$

from these we derive

26. $$\frac{d}{dq_i}\mathfrak{R}_i = -\frac{K_i^2}{\pi^2 q_i}\,\frac{d^2}{dv_i^2}\mathfrak{R}_i - \frac{1}{K_i}\,\frac{dK_i}{dq_i}\,v_i\frac{d}{dv_i}\mathfrak{R}_i.$$

We have also

$$q_i = e^{-\pi\frac{K_i'}{K_i}},$$

and therefore

27. $$\frac{1}{q_i}\,\frac{dq_i}{dk_i} = -\pi\,\frac{K_i\frac{dK_i'}{dk_i} - K_i'\frac{dK_i}{dk_i}}{K_i^2}$$
$$= -\frac{\pi}{k_ik_i'^2K_i^2}\{-K_iE_i' - K_i'E_i + K_iK_i'\}$$
$$= \frac{\pi^2}{2k_ik_i'^2K_i^2},$$

since $K_iE_i' + K_i'E_i - K_iK_i' = \frac{\pi}{2}$. Multiply 26 by $\frac{dq_i}{dk_i}$ and substitute from 27 and we have

$$\frac{d}{dk_i}\mathfrak{R}_i = -\frac{1}{2k_ik_i'^2}\,\frac{d^2}{dv_i^2}\mathfrak{R}_i - \frac{1}{K_i}\,\frac{dK_i}{dk_i}\,v_i\frac{d}{dv_i}\mathfrak{R}_i,$$

and hence Θ_4 which is the sum of the quantities $\mathfrak{R}_i$ satisfies the equation

$$\frac{d^2\theta_4}{dv_i^2} + \frac{2k_ik_i'^2}{K_i}\,\frac{dK_i}{dk_i}\,v_i\frac{d\theta_4}{dv_i} + 2k_ik_i'^2\frac{d\theta_4}{dk_i} = 0,$$

or

$$\frac{d^2\theta_4}{dv_i^2} - 2v_i\left(k_i'^2 - \frac{E_i}{K_i}\right)\frac{d\theta_4}{dv_i} + 2k_ik_i'^2\frac{d\theta_4}{dk_i} = 0.$$

For the constants K, k we have of course the following relations:

$$K_1 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2\varphi}}, \quad K_2 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_2^2 \sin^2\varphi}},$$

$$K_3 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_3^2 \sin^2\varphi}}, \quad K_4 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_4^2 \sin^2\varphi}},$$

or in the notation of hypergeometric series,

$$K_1 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_1^2), \quad K_2 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_2^2),$$

$$K_3 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_3^2), \quad K_4 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_4^2),$$

and for the constants E we have

$$E_1 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_1^2), \quad E_2 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_2^2),$$

$$E_3 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_3^2), \quad E_4 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_4^2).$$

Also for the K's

$$\sqrt{\frac{2K_1}{\pi}} = 1 + 2q_1 + 2q_1^4 + 2q_1^9 + 2q_1^{16} + \dots$$

$$\sqrt{\frac{2K_2}{\pi}} = 1 + 2q_2 + 2q_2^4 + 2q_2^9 + 2q_2^{16} + \dots$$

$$\sqrt{\frac{2K_3}{\pi}} = 1 + 2q_3 + 2q_3^4 + 2q_3^9 + 2q_3^{16} + \dots$$

$$\sqrt{\frac{2K_4}{\pi}} = 1 + 2q_4 + 2q_4^4 + 2q_4^9 + 2q_4^{16} + \dots$$

and

$$k_1 = \left\{ \frac{2q_1^{\frac{1}{4}} + 2q_1^{\frac{9}{4}} + 2q_1^{\frac{25}{4}} + 2q_1^{\frac{49}{4}} + \dots}{1 + 2q_1 + 2q_1^4 + 2q_1^9 + 2q_1^{16} + \dots} \right\}^2$$

$$k_2 = \left\{ \frac{2q_2^{\frac{1}{4}} + 2q_2^{\frac{9}{4}} + 2q_2^{\frac{25}{4}} + 2q_2^{\frac{49}{4}} + \dots}{1 + 2q_2 + 2q_2^4 + 2q_2^9 + 2q_2^{16} + \dots} \right\}^2$$

$$k_3 = \left\{ \frac{2q_3^{\frac{1}{4}} + 2q_3^{\frac{9}{4}} + 2q_3^{\frac{25}{4}} + 2q_3^{\frac{49}{4}} + \dots}{1 + 2q_3 + 2q_3^4 + 2q_3^9 + 2q_3^{16} + \dots} \right\}^2$$

$$k_4 = \left\{ \frac{2q_4^{\frac{1}{4}} + 2q_4^{\frac{9}{4}} + 2q_4^{\frac{25}{4}} + 2q_4^{\frac{49}{4}} + \dots}{1 + 2q_4 + 2q_4^4 + 2q_4^9 + 2q_4^{16} + \dots} \right\}^2.$$

We have seen that it is possible to derive a quadruple theta-function by operating upon the product of a single and of a triple theta-function, and generally that a p-tuple theta-function is derived by performing a certain operation upon the product of a single and of a p-1-tuple theta-function; and finally that the

p-tuple function can be obtained by operating upon the product of p single theta-functions; *i. e.* calling the operator ∇ we have

$$\vartheta\begin{pmatrix}\alpha_1\alpha_2\ldots\alpha_p\\ \beta_1\beta_2\ldots\beta_p\end{pmatrix}(v_1v_2\ldots v_p)=\nabla.\vartheta\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}(v_1)\vartheta\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}(v_2)\ldots\vartheta\begin{pmatrix}\alpha_p\\ \beta_p\end{pmatrix}(v_p).$$

It seems to me that it ought to be possible to obtain a theta-function of any order p by performing a proper operation upon certain combinations of theta-functions of *any* lower order. For example, cannot the quadruple function

$$\vartheta\begin{pmatrix}\alpha_1\alpha_2\alpha_3\alpha_4\\ \beta_1\beta_2\beta_3\beta_4\end{pmatrix}(v_1v_2v_3v_4)$$

be obtained by an operation performed upon a certain combination of the double theta-functions

$$\vartheta\begin{pmatrix}\alpha_1\alpha_2\\ \beta_1\beta_2\end{pmatrix}(v_1,v_2),\quad \vartheta\begin{pmatrix}\alpha_1\alpha_3\\ \beta_1\beta_3\end{pmatrix}(v_1,v_3),\quad \vartheta\begin{pmatrix}\alpha_1\alpha_4\\ \beta_1\beta_4\end{pmatrix}(v_1,v_4),$$

$$\vartheta\begin{pmatrix}\alpha_2\alpha_3\\ \beta_2\beta_3\end{pmatrix}(v_2,v_3),\quad \vartheta\begin{pmatrix}\alpha_2\alpha_4\\ \beta_2\beta_4\end{pmatrix}(v_2,v_4),\quad \vartheta\begin{pmatrix}\alpha_3\alpha_4\\ \beta_3\beta_4\end{pmatrix}(v_3,v_4)?$$

It is possible that some such expression may be known, but I have not seen it nor does it seem easy to obtain. It is, of course, perfectly simple to split up the right-hand members of equations 2 or 3 so that the double theta-function shall be brought in, but that reduction would obviously be of no value, as taking one further step we should arrive at an equation of the same form as 8. The question seems an interesting one and one worthy of investigation.

The theta-function under consideration may be written in the form

$$\begin{aligned}\vartheta\begin{pmatrix}\alpha_1\alpha_2\alpha_3\alpha_4\\ \beta_1\beta_2\beta_3\beta_4\end{pmatrix}(v_1v_2v_3v_4)&=\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}(-)^{\Sigma m\beta}q_1^{\frac{(2m_1+\alpha_1)^2}{4}}\ldots q_4^{\frac{2(m_4+\alpha_4)^2}{4}}\\ &\times q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}}\ldots q_{34}^{\frac{(2m_3+\alpha_3)(2m_4+\alpha_4)}{2}}\\ &\times e^{\frac{i\pi}{2K_1}(2m_1+\alpha_1)v_1+\ldots\frac{i\pi}{2K_4}(2m_4+\alpha_4)v_4}.\end{aligned}$$

The summations of course extend from $-\infty$ to $+\infty$ for all of the letters m_1, m_2, m_3, and m_4. This summation can be divided up into five parts: first, where all of the m's are zero, giving one term; second, where any three m's are zero while the fourth takes all values other than zero, giving four terms; third, where any two of the m's are zero and the other two take any values other than zero, giving six terms; fourth, where one of the m's is zero and the other three are not zero, giving four terms; and fifth, where none of the m's are zero, giving one term.

This gives us

$$\vartheta\binom{\alpha}{\beta}(v) = q_1^{\frac{\alpha_1^2}{4}}.q_2^{\frac{\alpha_2^2}{4}}.q_3^{\frac{\alpha_3^2}{4}}.q_4^{\frac{\alpha_4^2}{4}}.q_{12}^{\frac{\alpha_1\alpha_2}{2}}\dots q_{34}^{\frac{\alpha_3\alpha_4}{2}}.e^{\frac{i\pi}{2}\left[\frac{\alpha_1 v_1}{K_1}+\dots+\frac{\alpha_4 v_4}{K_4}\right]}$$

$$+\, q_2^{\frac{\alpha_2^2}{4}}.q_3^{\frac{\alpha_3^2}{4}}.q_4^{\frac{\alpha_4^2}{4}}.q_{23}^{\frac{\alpha_2\alpha_3}{2}}.q_{24}^{\frac{\alpha_2\alpha_4}{2}}.q_{34}^{\frac{\alpha_3\alpha_4}{2}}.e^{\frac{i\pi}{2}\left[\frac{\alpha_2 v_2}{K_2}+\dots+\frac{\alpha_4 v_4}{K_4}\right]}$$

$$\times \sum_{m_1=-\infty}^{m_1=+\infty}(-)^{m_1\beta_1} q_1^{\frac{(2m_1+\alpha_1)^2}{4}} q_{12}^{\frac{\alpha_2(2m_1+\alpha_1)}{2}}\dots q_{14}^{\frac{\alpha_4(2m_1+\alpha_1)}{2}}.e^{\frac{i\pi}{2K_1}(2m_1+\alpha_1)v_1}$$

$+$ (three similar terms)

$$+\, q_3^{\frac{\alpha_3^2}{4}}.q_4^{\frac{\alpha_4^2}{4}}.q_{34}^{\frac{\alpha_3\alpha_4}{2}}.e^{\frac{i\pi}{2}\left[\frac{\alpha_3 v_3}{K_3}+\frac{\alpha_4 v_4}{K_4}\right]}$$

$$\times \sum_{m_1=-\infty}^{m_1=\infty}\sum_{m_2=-\infty}^{m_2=\infty}(-)^{m_1\beta_1+m_2\beta_2}.q_1^{\frac{(2m_1+\alpha_1)^2}{4}}.q_2^{\frac{2(m_2+\alpha_2)^2}{4}}\dots q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}}.q_{13}^{\frac{\alpha_3(2m_1+\alpha_1)}{2}}$$

$$\times\, q_{14}^{\frac{\alpha_4(2m_1+\alpha_1)}{2}}.q_{23}^{\frac{\alpha_3(2m_2+\alpha_2)}{2}}.q_{24}^{\frac{\alpha_4(2m_2+\alpha_2)}{2}}.e^{\frac{i\pi}{2}\left[\frac{(2m_1+\alpha_1)v_1}{K_1}+\frac{(2m_2+\alpha_2)v_2}{K_2}\right]}$$

$+$ (five similar terms) $+$ etc.

The remaining terms are formed in the manner indicated above, but it is not worth while writing them down, as the formula is too complicated to deal with in the general case. There is one class of cases, or rather one group of functions, sixteen in number, for which this formula may be very much simplified—these are the functions for which all of the indices α are zero. The five sets of terms in the general formula correspond to the following values of the α's.

	α_1	α_2	α_3	α_4
I	0	0	0	0
II	1	0	0	0
	0	1	0	0
	0	0	1	0
	0	0	0	1
III	1	1	0	0
	0	1	1	0
	0	0	1	1
	1	0	0	1
	1	0	1	0
	0	1	0	1
IV	1	1	1	0
	0	1	1	1
	1	0	1	1
	1	1	0	1
V	1	1	1	1

Combining each of these 16 groups of the values of $\alpha_1 \ldots \alpha_4$ with the corresponding 16 groups for the values of $\beta_1 \ldots \beta_4$ we have the whole 256 functions. The simplest way of representing each of these functions by a formula similar to the general one given above would be to work it out *ab initio*, assuming the particular characteristic and developing for that special case, but a table of that magnitude could hardly have any practical value. We may take, however, the case where

$$(\alpha_1 \alpha_2 \alpha_3 \alpha_4) = (0\,0\,0\,0)$$

and give the β-row all of its values—we thus have the sixteen functions referred to above. Substitute these values of the α's in the general formula and make also

$$(\beta_1 \beta_2 \beta_3 \beta_4) = (0\,0\,0\,0)$$

then we have

$$\vartheta\begin{pmatrix}0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}(v_1,\, v_2,\, v_3,\, v_4) =, \text{ say } \vartheta_0(v) =$$

$$1 + 2\sum_{m_1=1}^{m_1=\infty} q_1^{m_1^2} \cos\frac{m_1 \pi v_1}{K_1} + \ldots + 2\sum_{m_4=1}^{m_4=\infty} q_4^{m_4^2} \cos\frac{m_4 \pi v_4}{K_4}$$

$$+ \sum_{m_1=-\infty}^{m_1=+\infty} \sum_{m_2=-\infty}^{m_2=+\infty} q_1^{m_1^2} q_2^{m_2^2} q_{12}^{2m_1 m_2} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2}\right]} + \text{(five similar terms)}$$

$$+ \sum_{m_1=-\infty}^{m_1=+\infty} \sum_{m_2=-\infty}^{m_2=+\infty} \sum_{m_3=-\infty}^{m_3=+\infty} q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2} q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3}\right]}$$

$$+ \text{(three similar terms)}$$

$$+ \sum_{m_1=-\infty}^{m_1=+\infty} \sum_{m_2=-\infty}^{m_2=+\infty} \sum_{m_3=-\infty}^{m_3=+\infty} \sum_{m_4=-\infty}^{m_4=+\infty} q_1^{m_1^2} \ldots q_4^{m_4^2} . q_{12}^{2m_1 m_2} \ldots q_{34}^{2m_3 m_4} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \ldots \frac{m_4 v_4}{K_4}\right]}.$$

In this last term the values m_1, m_2, m_3, $m_4 = 0$ are excluded.

The first line in the right-hand member of this equation is, in the ordinary notation,

$$\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_4) - 3.$$

The first term in the group of double summations is the double theta-function

$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1 v_2).$$

Transforming this in the same manner we have (see Forsyth's memoir, page 809)

$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1 v_2) = 1 + 2\sum_{m_1=1}^{m_1=\infty} q_1^{m_1^2} \cos\frac{m_1 \pi v_1}{K_1} + 2\sum_{m_2=1}^{m_2=\infty} q_2^{m_2^2} \cos\frac{m_2 \pi v_2}{K_2}$$

$$+ 2\sum_{m_1=1}^{m_1=\infty} \sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2} \left\{ q_{12}^{2m_1 m_2} \cos\pi\left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2} \cos\pi\left(\frac{m_1 v_1}{K_1} - \frac{m_2 v_2}{K_2}\right) \right\}$$

or
$$\vartheta\begin{pmatrix}00\\00\end{pmatrix}(v_1 v_2) = \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) - 1$$
$$+ 2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2}\left\{ q_{12}^{2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}-\frac{m_2 v_2}{K_2}\right)\right\}.$$

The remaining five terms under the double summation are at once obtained from this.

Take now the terms under the triple summation sign. The first of these is
$$\vartheta\begin{pmatrix}000\\000\end{pmatrix}(v_1 v_2 v_3).$$

We reduce this just as we reduced the quadruple function, and we have
$$\vartheta\begin{pmatrix}000\\000\end{pmatrix}(v_1 v_2 v_3) = 1 + 2\sum_{m_1=1}^{m_1=\infty} q_1^{m_1^2}\cos\frac{m_1\pi v_1}{K_1} + \ldots + 2\sum_{m_3=1}^{m_3=\infty} q_3^{m_3^2}\cos\frac{m_3 \pi v_3}{K_3}$$
$$+ \sum_{m_1=-\infty}^{m_1=\infty}\sum_{m_2=-\infty}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2} q_{12}^{2m_1 m_2} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right]} + \text{(two similar terms)}$$
$$+ \sum_{m_1=-\infty}^{m_1=\infty}\sum_{m_2=-\infty}^{m_2=\infty}\sum_{m_3=-\infty}^{m_3=\infty} q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2} q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}+\frac{m_3 v_3}{K_4}\right]}.$$

In this last term the values $m_1, m_2, m_3 = 0$ are excluded.

The first five terms on the right-hand side of this equation are
$$= \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3) - 2.$$

The terms under the double summation sign are
$$\vartheta\begin{pmatrix}00\\00\end{pmatrix}(v_1 v_2) = \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) - 1$$
$$+ 2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2}\left\{ q_{12}^{2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}-\frac{m_2 v_2}{K_2}\right)\right\}$$

and two similar ones. Collecting together all our results we have now
$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(v_1 v_2 v_3 v_4) = 7\left\{\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3)\right\} + 4\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_4) - 14 + \mathfrak{A}.$$

Here
$$\mathfrak{A} = 2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2}\left\{ q_{12}^{2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}-\frac{m_2 v_2}{K_2}\right)\right\}$$
$$+ \text{(five similar terms)}$$

$$+2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2}\left\{ q_{12}^{2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right)+q_{12}^{-2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}-\frac{m_2 v_2}{K_2}\right)\right\}$$

+ (eleven similar terms)

$$+\sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=-\infty}^{\infty}{}' q_1^{m_1^2}\ldots q_3^{m_3^2} q_{12}^{2m_1 m_2}\ldots q_{23}^{2m_2 m_3} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\ldots+\frac{m_3 v_3}{K_4}\right]}$$

$$+\sum_{m_1=-\infty}^{\infty}{}' \sum_{m_2=-\infty}^{\infty}{}' \sum_{m_3=-\infty}^{\infty}{}' \sum_{m_4=-\infty}^{\infty}{}' q_1^{m_1^2}\ldots q_4^{m_4^2} q_{12}^{2m_1 m_2}\ldots q_{34}^{2m_3 m_4} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\ldots+\frac{m_4 v_4}{K_4}\right]}.$$

The accented Σ means that the values $m=0$ are to be left out in forming the sum.

We proceed now to a more complete reduction of the quantity $\mathfrak{A}$. The terms under the double summation sign require only to be added together, no further reduction being necessary. The terms under each of the triple summation signs divide up into four groups, viz., taking the first case where the summations refer to m_1, m_2 and m_3 we have

I.	m_1, m_2, m_3 all positive	One term.
II.	Any two of these positive and the third negative .	Three terms.
III.	Any one positive and the remaining two negative .	Three terms.
IV.	All three negative	One term.
	In all	Eight terms.

For the quadruple summations we have similarly sixteen terms. Taking now the case of the triple summations and we find the following values, in which for convenience I have written

$$Q_4 = q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2}, \quad u=\frac{v}{K}\cdot$$

I. m_1, m_2, m_3 all positive.

$$\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3}\,.\, e^{i\pi(m_1 u_1+m_2 u_2+m_3 u_3)}.$$

II. Any two positive and the third negative.

$$\left\{\begin{array}{l}\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{2m_2 m_3}\,.\, e^{i\pi(-m_1 u_1+m_2 u_2+m_3 u_3)} \\ \sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{-2m_2 m_3}\,.\, e^{i\pi(m_1 u_1-m_2 u_2+m_3 u_3)} \\ \sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{-2m_2 m_3}\,.\, e^{i\pi(m_1 u_1+m_2 u_2-m_3 u_3)}\end{array}\right.$$

III. Any two negative and the third positive.

$$\left\{\begin{array}{l}\sum\limits_1^\infty\sum\limits_1^\infty\sum\limits_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{-2m_2m_3}\,.\,e^{i\pi(-m_1u_1-m_2u_2+m_3u_3)}\\ \sum\limits_1^\infty\sum\limits_1^\infty\sum\limits_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{-2m_2m_3}\,.\,e^{i\pi(-m_1u_1+m_2u_2-m_3u_3)}\\ \sum\limits_1^\infty\sum\limits_1^\infty\sum\limits_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{2m_2m_3}\,.\,e^{i\pi(m_1u_1-m_2u_2-m_3u_3)}.\end{array}\right.$$

IV. All three negative.

$$\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{2m_2m_3}\,.\,e^{-i\pi(m_1u_1+m_2u_2+m_3u_3)}.$$

The summations are of course for m_1, m_2 and m_3.

Adding groups I and IV and we have obviously

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right).$$

The first of group II added to the third of group III gives

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right).$$

The second of II added to the second of III gives

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right).$$

The third of II added to the first of III gives

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}\right).$$

Combining all of these we have for the first term under the triple summation sign

$$\begin{aligned}&2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)\\ &+2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)\\ &+2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)\\ &+2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}\right).\end{aligned}$$

The remaining three terms are of course obtained by replacing the suffixes (1, 2, 3) by (2, 3, 4), (3, 4, 1) and (4, 1, 2).

Take up now the quadruple summation. As already remarked there are, in this case, sixteen groups of terms, these combine however into eight. The process of reduction is exactly the same as in the case of the triple summation,

so it is not necessary to go further into it. We have, in fact, for the quadruple summation the following value, where for brevity I have written

$$Q = q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2} q_4^{m_4^2},$$

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty Q\, q_{12}^{2m_1m_2}\dots q_{34}^{2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+\,2\sum\sum\sum\sum Q\, q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$

The limits of the summations in the last seven terms are of course the same as in the first term. We have now merely to add together all of our results in order to obtain the value of the quantity $\mathfrak{A}$. Notice that in the above given value of $\mathfrak{A}$ of the "*(eleven similar terms)*" one is identical in form and value with the one written down, and the remaining ten terms reduce to five each of which is repeated once. We have then finally

$$\vartheta\binom{0000}{0000}(v_1v_2v_3v_4) = 7\left\{\vartheta\binom{0}{0}(v_1)+\vartheta\binom{0}{0}(v_2)+\vartheta\binom{0}{0}(v_3)\right\} + 4\vartheta\binom{0}{0}(v_4) - 14$$
$$+\,4\sum_1^\infty\sum_1^\infty q_1^{m_1^2} q_2^{m_2^2}\left\{q_{12}^{2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right)+q^{-2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}\right)\right\}$$
$+$ (five similar terms)
$$+\,2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4\left\{q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)\right.$$
$$+\,q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{2m_2m_3}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)$$
$$+\,q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)$$
$$\left.+\,q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}\right)\right\}$$
$+$ (three similar terms)

$$+ 2\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty Q \left\{ q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)\right.$$
$$+ q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+ q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+ q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+ q_{12}^{2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$
$$+ q_{12}^{2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$+ q_{12}^{-2m_1m_2} q_{13}^{2m_1m_3} q_{14}^{-2m_1m_4} q_{23}^{-2m_2m_3} q_{24}^{2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$
$$\left.+ q_{12}^{-2m_1m_2} q_{13}^{-2m_1m_3} q_{14}^{2m_1m_4} q_{23}^{2m_2m_3} q_{24}^{-2m_2m_4} q_{34}^{-2m_3m_4} \cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)\right\}.$$

In this formula and in those preceding it is of course not strictly accurate to write the quadruple function in the form

$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(v_1v_2v_3v_4);$$

it really should be written

$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(u_1u_2u_3u_4),$$

but as the quantities v_1, v_2, v_3, v_4 appear in the right-hand members of the equations I have written them in the left-hand members also. The equations connecting the v's and the u's are as above

$$K_1u_1=v_1,\quad K_2u_2=v_2,\quad K_3u_3=v_3,\quad K_4u_4=v_4.$$

The remaining 255 quadruple theta-functions may be thrown into forms similar to the one just examined, but as the formulæ are so very long and as the process has been already sufficiently indicated, it is not worth while to work out any more of them.

It would not be difficult to extend the above processes to the case of p-tuple theta-functions, but it is not the object of the present paper to do so. Still it may be remarked that the term in the p-tuple theta-function corresponding to the term $\mathfrak{A}$ just computed will consist of a group of terms involving double summations, another group involving triple summations, and so on until we come to a single term which involves a p-tuple summation. The method of computing

these terms is identical with that above given for the quadruple functions. Take for example $r < p$, then the r-tuple summations are divided up as follows: suppose first r even, say $r = 2\rho$.

		NUMBER OF CASES.
$(1)_1$	All the m's positive, ...	1
$(1)_2$	All the m's but one positive, .	2ρ
$(1)_3$	All the m's but two positive, .	$\frac{2\rho(2\rho-1)}{2}$
. . . . .	. . . .	. . . .
$(1)_{\rho+1}$	All the m's but ρ positive, . .	$\frac{2\rho(2\rho-1)(2\rho-2)\ldots(\rho+1)}{\rho}$
. . . . .	. . . .	. . . .
$(1)_{2\rho-1}$	All the m's negative but two .	$\frac{2\rho(2\rho-1)}{2}$
$(1)_{2\rho}$	All the m's negative but one .	2ρ
$(1)_{2\rho+1}$	All the m's negative, ...	1

Here the first and last terms are to be combined, also the second and the last but one, and so on until we come to the middle term; this term is made up of

$$\frac{2\rho(2\rho-1)(2\rho-2)\ldots\rho+1}{\rho}$$

(an even number) of simple terms, and the first half of these simple terms is to be combined with the second half in order to obtain forms similar to those given in the case of the quadruple theta-functions. The case where r is odd follows at once from the case of r even.